Section 6.2: Volumes

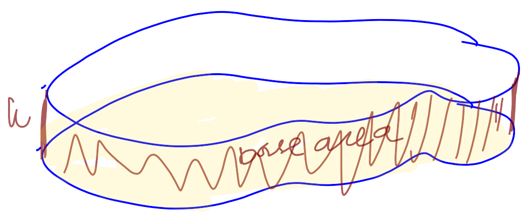
Objective: In this lesson, you learn

- ☐ How to establish the volume of a solid of revolution as the limit of a Riemann sum.

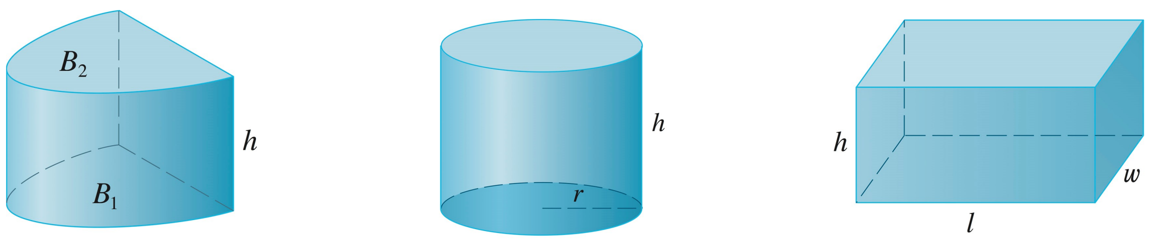
I. Volumes

The volume of a cylindrical solid is always defined to be its base area times its height:

$$\textbf{Volume} = \textbf{Area} \times \textbf{Height}$$



Some classic formulas: A solid S that is a right cylinder (a known base area A and height h).

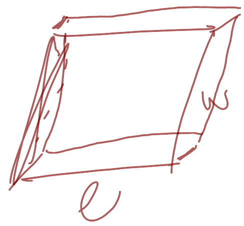


(a) Cylinder $V = Ah$ (b) Circular cylinder $V = \boxed{\pi r^2}h$ (c) Rectangular box $V = lwh$

The area of a circle

Problem: How do we define the volume of a solid S that is not a right cylinder?

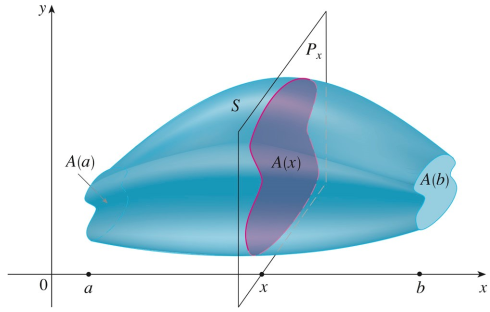
The base Area and height are not know.



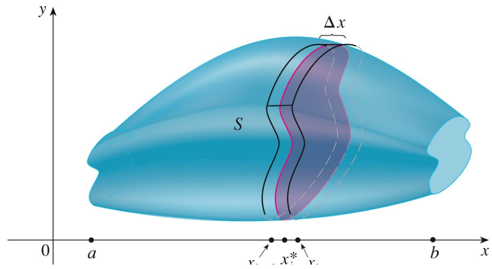
Volumes

For a solid S that is not a right cylinder, we "cut" into pieces and approximate each piece by a cylinder. Then the volume is estimated by adding the volumes of the cylinders.

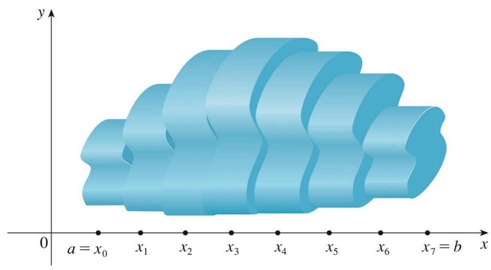
1. Start by intersection S with a plane region that is called a **cross-section** of S.
2. Let $A(x)$ be the area of the cross-section of S in a plane P_x cutting S perpendicularly to the $x-$axis and passing through the point x, where $a \leq x \leq b$.



3. Divide S into n "slabs" of equal width $\triangle x$ by using the planes $P_{x_1}, P_{x_2}, \cdots, P_{x_n}$ to slice the solid.
4. If we choose samples points x^* in $[x_{i-1}, x_i]$, we can approximate the i^{th} slab S_i by a cylinder with base area $A(x_i^*)$ and height $\triangle x$.
5. The volume of this cylinder is $A(x_i^*)\triangle x$.



6. Add the volumes of the slabs to approximate the total volume, V, which is the limit of
$$\sum_{i=1}^{n} A(x_i^*)\triangle x \ .$$
7. The approximation gets better as $n \to \infty$.



So we can define the volume of a solid as follows:

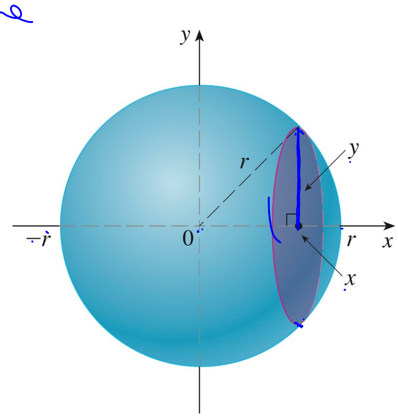
Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through S and perpendicular to the x-axis, is $A\,(x)$, where A is a continuous function, then the volume of S is
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A\,(x_i^*)\,\Delta x = \int_a^b A\,(x)\,dx.$$

Remark: Note that in the volume formula $V = \int_a^b A\,(x)dx$, $A\,(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Example 1: Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$

A point x, what is the radius of the cross-section?

the radius is y, what is y?



$$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$$

$$\Rightarrow y = \pm\sqrt{r^2 - x^2}$$

$$\boxed{y = +\sqrt{r^2 - x^2}}$$

So, $A(x)$ the area of the cross-section is (circle)

$$A(x) = \pi y^2 = \pi \cdot \left(\sqrt{r^2 - x^2}\right)^2$$

$$\boxed{A(x) = \pi \cdot (r^2 - x^2)}$$

The volume is

$$V = \int_{-r}^{r} A(x)\, dx = \int_{-r}^{r} \pi (r^2 - x^2)\, dx$$

$$= 2\pi \int_0^r r^2 - x^2\, dx \qquad \text{since } (r^2 - x^2 \text{ is an even function})$$

$$= 2\pi \left(r^2 x - \frac{x^3}{3} \right\Big|_0^r$$

$$= 2\pi \left[\left(r^2 \cdot (r) - \frac{r^3}{3} \right) - \cancel{\left(r^2 \cdot (0) - \frac{0^3}{3} \right)} \right] \quad 0$$

$$= 2\pi \left(\frac{3}{3} r^3 - \frac{r^3}{3} \right)$$

$$= 2\pi \left(\frac{3r^3 - r^3}{3} \right) = 2\pi \left(\frac{2r^3}{3} \right)$$

$$\boxed{V = \frac{4}{3}\pi r^3}$$

$y=\sqrt{x}$

$x=y^2$

II. Solids of revolution

Solids of revolution are obtained by revolving a region about a line.

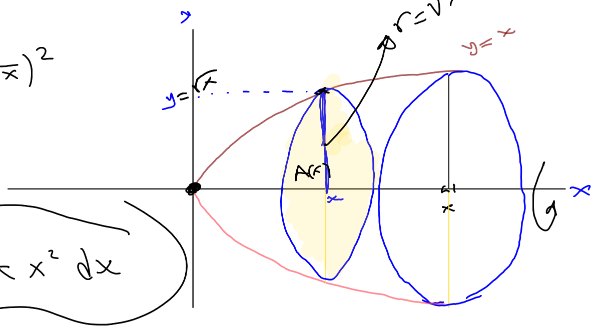
$f(x)=\sqrt{x}$

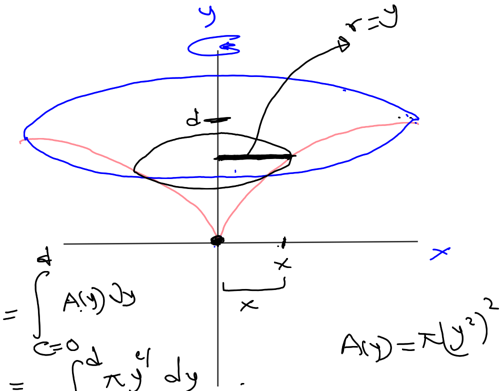
$A(x)=\pi(\sqrt{x})^2$

$=\pi x^2$

$\int_a^b A(x)\,dx$

$V=\int_a^b \pi x^2\,dx$





$V=\int_{c=0}^{d} A(y)\,dy$

$=\int_0^d \pi y^4\,dy$

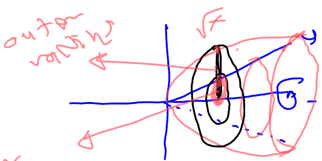
$A(y)=\pi(y^2)^2$

Compute the volume of a solid of revolution by using the basic formula $V = \int_a^b A(x)\,dx$ or $V = \int_c^d A(y)\,dy$, and find the cross-sectional area $A(x)$ or $A(y)$ in one of the following ways:

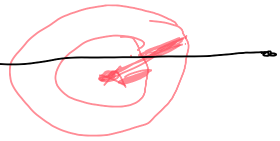
i. If the cross-section is a disk, find the radius of the disk (in terms of x or y) and use

$$A = \pi(\text{radius})^2.$$

ii. If the cross-section is a washer, find the inner radius and outer radius and use



$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2.$$

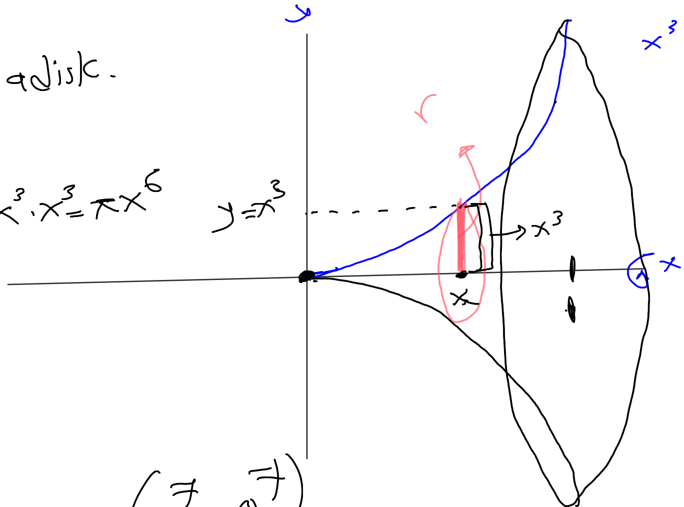


Example 2: Calculate the volume of the solid, obtained by rotating the curve $y = x^3$ for $0 \le x \le 1$ around the x-axis?

① The cross-section is a disk.

② Radius $r=x^3$

③ $A=\pi(r)^2=\pi(x^3)^2=\pi x^3\cdot x^3=\pi x^6$



$$V=\int_0^1 A(x)\,dx=\int_0^1 \pi x^6\,dx$$

$$=\pi \frac{x^7}{7}\Big|_0^1=\frac{\pi}{7}(1^7-0^7)$$

$$=\boxed{\pi/7}\cong 0.449$$

Example 3: Rotate the same curve, in example 2, around the y-axis? $\rightarrow \int A(y)\, dy$

[1] The cross-section is a disk.

[2] The Radius $r = y^{1/3} = \sqrt[3]{y}$

[3] $A = \pi (r)^2 = \pi \cdot \left(y^{1/3}\right)^2 = \pi y^{2 * \frac{1}{3}}$

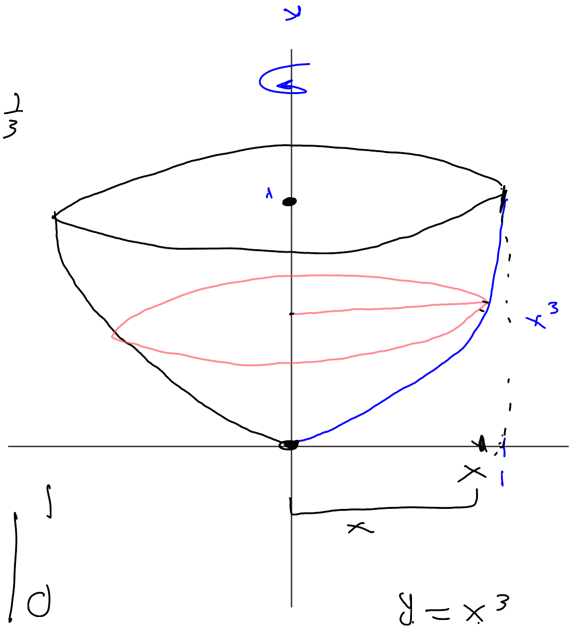
$$\boxed{A(y) = \pi y^{2/3}}$$

(4) $V = \int_{y=0}^{y=1} A(y)\, dy$

$= \int_{y=0}^{y=1} \pi y^{2/3}\, dy = \pi \frac{3}{5} y^{5/3} \Big|_0^1$

$= \frac{3\pi}{5}\left(1^{5/3} - 0^{5/3}\right)$

$= \boxed{\frac{3\pi}{5}} \approx 1.885$



$y = x^3$

$x = \sqrt[3]{y}$

$x = 0 \Rightarrow y = 0^3 = 0$

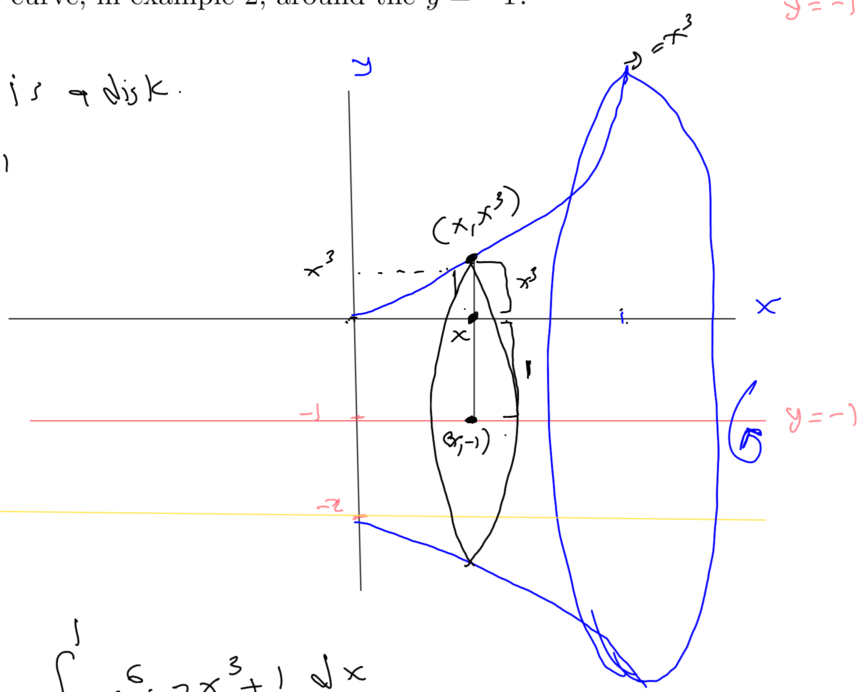
$x = 1 \Rightarrow y = 1^3 = 1$

Example 4: Rotate the same curve, in example 2, around the $y = -1$?

[1] The cross-section is a disk.

[2] The Radius $r = x^3 + 1$

[3] $A(x) = \pi (x^3 + 1)^2$



$V = \int_0^1 A(x)\, dx$

$= \int_0^1 \pi (x^3+1)^2\, dx$

$= \pi \int_0^1 (x^3)^2 + 2 \cdot x^3 \cdot 1 + 1^2\, dx = \pi \int_0^1 x^6 + 2x^3 + 1\, dx$

$= \pi \left(\frac{x^7}{7} + \frac{2}{4} x^4 + x \Big|_0^1 \right) = \pi\left(\frac{1}{7} + \frac{1}{2} + 1\right)$

$(a+b)^2 = a^2 + 2ab + b^2$

$= \pi \frac{2+7+14}{14} = \frac{23\pi}{14}$

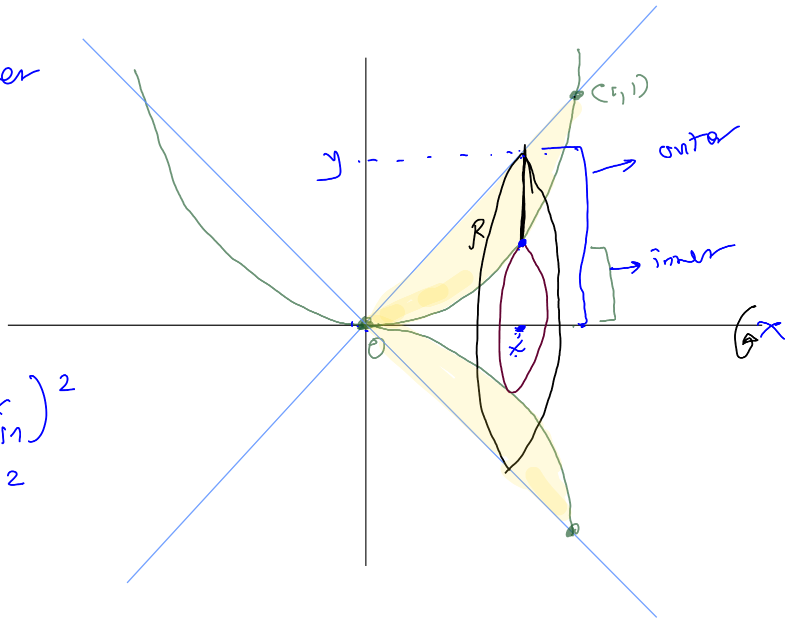
Example 5: Calculate the ~~volume of~~ the solid, obtained by rotating the region R around the $x-$axis, where the region R is enclosed by the curves $y = x^2$ and $y = x$?

1. The cross-section is a washer

2. The radius is

outer radius $r_{out} = x$

$r_{in} = x^2$



3. $A(x) = \pi (r_{out})^2 - \pi (r_{in})^2$

$= \pi (x)^2 - \pi (x^2)^2$

$= \pi [x^2 - x^4]$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi (x^2 - x^4)\,dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

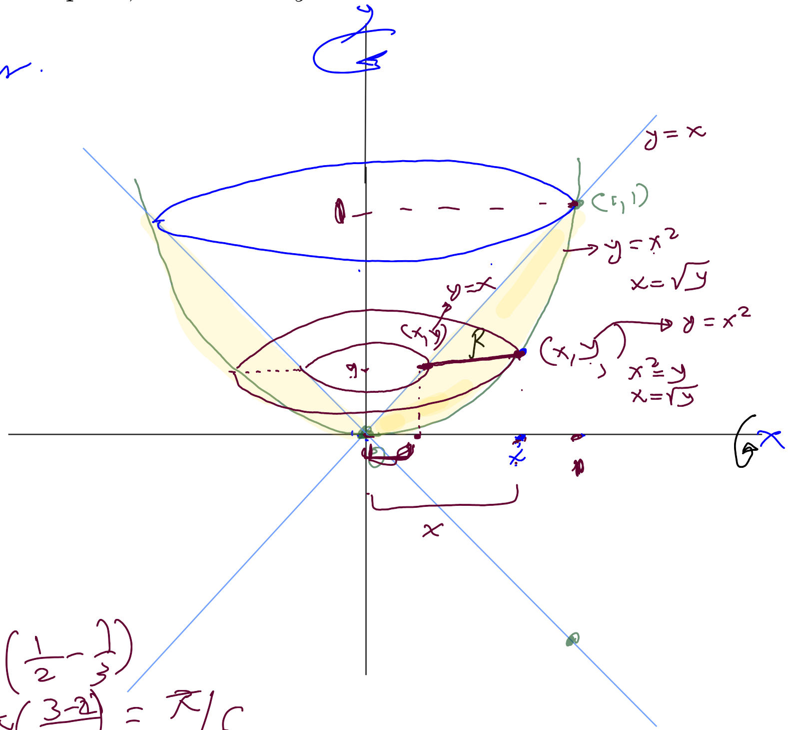
$$= \pi \left[\left(\frac{1}{3} - \frac{1}{5} \right) - (0 - 0) \right]$$

$$= \pi \left(\frac{5 - 3}{15} \right)$$

$$= \boxed{\frac{2\pi}{15}}$$

Example 6: Rotate the same curve, in example 5, around the $y-$axis?

① The cross-section is a washer.



$\boxed{2}$ $r_{out} = \sqrt{y}$

$r_{in} = y$

$\boxed{3}$ $A(y) = \pi (r_{out})^2 - \pi (r_{in})^2$

$= \pi (\sqrt{y})^2 - \pi (y)^2$

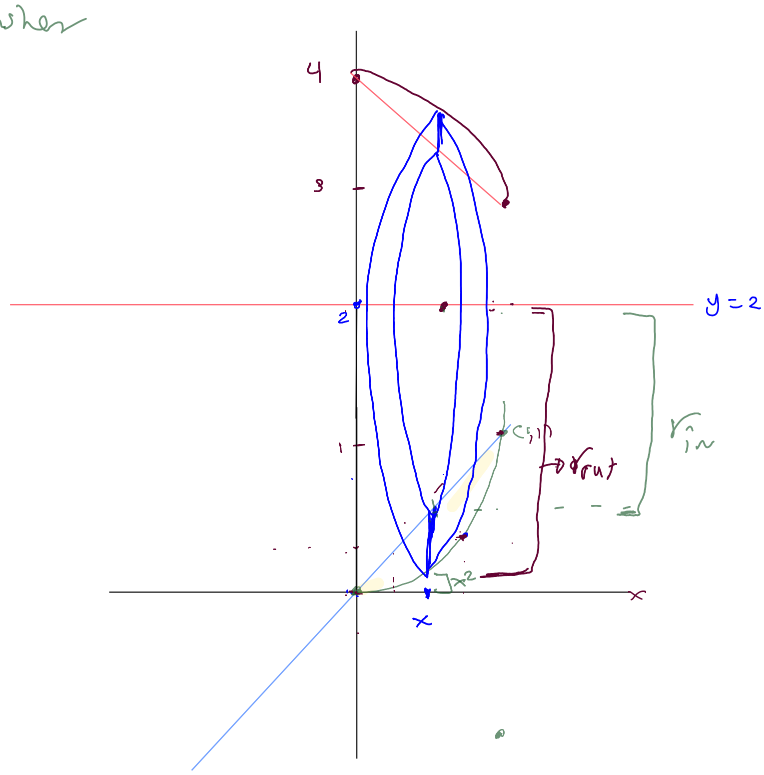
$$V = \int_0^1 A(y)\, dy = \int_0^1 \pi y - \pi y^2 \, dy$$

$$= \pi \left(\frac{y^2}{2} - \frac{y^3}{3} \Big|_0^1 \right) = \pi \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \pi \left(\frac{3-2}{6} \right) = \pi/6.$$

Example 7: Rotate the same curve, in example 5, around the $y = 2$?

1: The cross-section is a washer



Homework: Rotate the same curve, in example 5, around the $x = -1$ or $x = 2$?